

SENIOR THESIS

By

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Abstract

The problem of monitoring an electric power system by placing phase measuring units can be studied as a power dominating set problem in graph theory. This problem is closely related to the well-known domination problem in graphs, and is a close cousin of the well-studied zero-forcing set. A set S of vertices is defined to be a power dominating set of a graph if every vertex and every edge in the graph is monitored by the set S (following a set of rules for power system monitoring). The minimum cardinality of a power dominating set of a graph is its power domination number. In this thesis, we present upper bounds on the power domination numbers for general Cartesian Products of two graphs as well as Cartesian products of three graphs of the same classes of graphs. We also present a global upper bound for cubic graphs and the power domination number for Circulant Graphs. The power domination numbers for Mycielski Graphs are also characterized.

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1 Introduction

Electric power companies need to continually monitor their system's state as defined by a set of state variables. One method of monitoring these variables is to place phase measurement units (PMUs) at selected locations in the system. Because of the high cost of a PMU, it is desirable to minimize the number of PMUs while maintaining the ability to monitor (observe) the entire system. A system is said to be observed if all of the state variables of the system can be determined from a set of measurements (e.g., voltages and currents).

This optimization problem can be modeled as a power domination problem where the electrical grid is represented by a graph. Let $G = (V(G), E(G))$ be a graph, then $V(G)$ includes possible locations where PMUs can be located, and $E(G)$ denotes the set of wires carrying electricity. A set $S \subseteq V(G)$ is said to be a power dominating set (PDS) if every vertex and edge in G are observed by S according to the following observation rules based on Ohm's Law and Kirchhoff's current law:

1. Any vertex a PMU is placed and its incident edges are observed.
2. If one end vertex of an observed edge is observed, then the other end vertex is observed.
3. Any edge connecting two observed vertices is observed.
4. If a vertex is of degree $k > 1$, and $k - 1$ of these incident edges are observed, then all k incident edges are observed.

1.1 Definitions and Terminology

Denote the cardinality of a set S as $|S|$. We define a graph $G = (V(G), E(G))$ such that $V(G)$ denotes the set of vertices of G and $E(G)$ denotes the set of edges between the vertices of G . The edge $e = \{x, y\} \in E$ is frequently abbreviated as either e or xy . A multigraph is a graph that can have more than one edge between a pair of vertices. A vertex $w \in V(G)$ is said to be a neighbor of or adjacent to a vertex $v \in V(G)$ if $vw \in E(G)$. The *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = \{w \in V(G) \mid vw \in E(G)\}$. The closed neighborhood of $v \in V(G)$ is $N_G[v] = N_G(v) \cup \{v\}$. The neighborhood of a vertex set S , $N_G[S] = \bigcup_{s \in S} N_G[s]$. The *degree* of a vertex v in G , denoted $\deg_G(v)$, is defined as $\deg_G(v) = |N_G(v)|$. A 2-factor of graph G is a set of vertex disjoint cycles that span G .

A set $S \subset V(G)$ is a *dominating set* for G if for each $v \in V(G)$, the vertex $v \in S$, or is adjacent to a member of S . If S is a dominating set, and $u \in S$ is adjacent to $v \in V(G) - S$, then we say u observes, or dominates v . If the cardinality of a dominating set S is minimum, then S is called a minimum dominating set and $|S|$ is called the domination number of G , denoted by $\gamma(G)$.

A *power dominating set* of a graph $G = (V(G), E(G))$ is a set $S \subset V(G)$ that observes every vertex of G according to the following rules: in the first timestep, every vertex in $N[S]$ is observed; in each subsequent timestep, every vertex which is the only unobserved neighbor of some observed vertex becomes observed. The first timestep is called the *domination step* and each subsequent timestep is called a *forcing step*. Given a power dominating set S , we let S_i represent the vertices that are observed by timestep i . Thus, $N[S] = S_1$ and S_1 is the set of vertices observed in the domination step. As we would prefer to minimize the number of vertices needed in a power dominating set, we define, the power domination number of G , denoted by $\gamma_p(G)$, is the minimum cardinality among all power dominating sets of G .

Algorithm [4]:

Let $S \subseteq V(G)$ be the set of vertices where the PMUs are placed.

1. (Domination)

Set $M(S) \leftarrow S \cup N(S)$

2. (Propagation)

As long as there exists $v \in M(S)$ such that $N(v) \cap (V(G) - M(S)) = \{w\}$, set $M(S) \leftarrow M(S) \cup \{w\}$.

1.2 Known Bounds and Equalities for γ_p

In this section, known results for the bounds on the power domination number and for some families of graphs will be discussed.

Theorem 1. [3] For any graph G , $1 \leq \gamma_P(G) \leq \gamma(G)$.

In [3], the authors present many observations on the power domination number for common families of graphs and we recall some of them here. Let K_n, P_n, W_n and C_n denote, respectively, the complete graph, path, wheel and cycle of order n . Also, let $K_{1,n}$ denote the star with $n + 1$ vertices such that n of them are end-vertices.

Theorem 2. [3] For the graph G where $G \in \{K_n, C_n, P_n, K_{1,n}, K_{2,n}\}$, $\gamma_P(G) = 1$

Theorem 3. [3] For any tree T , $\gamma_P(T) = 1$ if and only if T is homeomorphic to the graph $K_{1,n}$ for some positive integer n .

A spider is a tree with at most one vertex having degree 3 or more. The spider number of a tree T , denoted by $sp(T)$, is the minimum number of subsets into which $V(T)$ can be partitioned so that each subset induces a spider. We call such a partition a spider partition and each set of the partition a spider subset.

Theorem 4. [3] For any tree T , $sp(T) = \gamma_P(T)$

2 Cartesian products

The *Cartesian product* $G \square H$ of graphs G and H is the graph with vertex set $V(G \square H) = V(G) \times V(H)$ whereby two vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G \square H$ if either $x_1 = x_2$ and $y_1 y_2 \in E(H)$ or $y_1 = y_2$ and $x_1 x_2 \in E(G)$. For a vertex $g \in V(G)$, the subgraph of $G \square H$ induced by the set $\{(g, h) \mid h \in V(H)\}$ is called a H -layer and is denoted H^g . Similarly, for a vertex $h \in V(H)$, the subgraph induced of $G \square H$ induced by the set $\{(g, h) \mid g \in V(G)\}$ is called a G -layer and is denoted G^h .

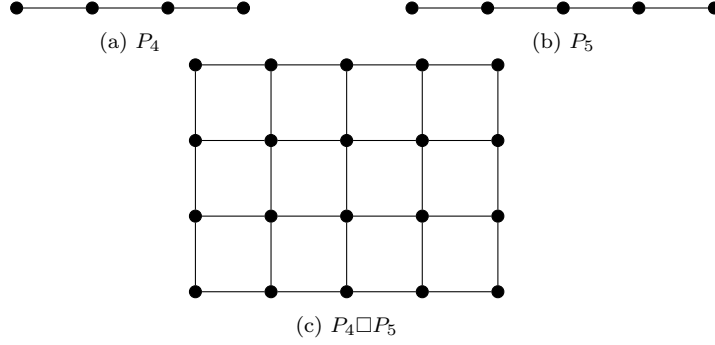


Figure 1: The Cartesian Product of P_4 and P_5

The power domination number of the Cartesian products of two graphs G and H has been well studied when both G and H are either a path, cycle, star, wheel, or complete graph. Notably, the following has been shown.

Theorem 5. [2] For $1 \leq n \leq m$,

$$\gamma_P(P_m \square P_n) = \begin{cases} \lceil \frac{n+1}{4} \rceil & \text{if } m \equiv 4 \pmod{8} \\ \lceil \frac{n}{4} \rceil & \text{otherwise} \end{cases}$$

Theorem 6. [1] For any positive integers m and n , $\gamma_P(P_n \square C_m) \leq \lceil \frac{m+1}{4} \rceil$.

Theorem 7. [4] For $m \geq n \geq 3$, $\gamma_P(K_{1,m} \square K_{1,n}) = n - 1$.

We now give a general upper bound on $\gamma_P(G \square H)$ for any two graphs G and H .

Theorem 8. For any graphs G and H ,

$$\gamma_P(G \square H) \leq \min\{\gamma_P(G)|V(H)|, \gamma_P(H)|V(G)|, \gamma(G)(|V(H)|-1), \gamma(H)(|V(G)|-1)\}.$$

Proof. We first show that $\gamma_P(G \square H) \leq \min\{\gamma_P(G)|V(H)|, \gamma_P(H)|V(G)|\}$. Let D be a power dominating set of G and consider the set $S = D \times V(H)$. Suppose $u \in V(G)$ is observed by $y \in V(G)$ at timestep t . That is, u is the only unobserved neighbor of

y at timestep $t - 1$. For any $h \in V(H)$, (u, h) will be the only unobserved neighbor of (y, h) at timestep $t - 1$ for each $h \in V(H)$. Thus, S will in fact power dominate $G \square H$.

We will extend this upper bound to

$$\gamma_P(G \square H) \leq \min\{\gamma_P(G)|V(H)|, \gamma_P(H)|V(G)|, \gamma(G)(|V(H)| - 1), \gamma(H)(|V(G)| - 1)\}.$$

Our claim is that $G \square H$ can always be power dominated by vertices of order

$$\min\{\gamma(G)(|V(H)| - 1), \gamma(H)(|V(G)| - 1)\}.$$

Without the loss of generality, assume that

$$\min\{\gamma(G)(|V(H)| - 1), \gamma(H)(|V(G)| - 1)\} = \gamma(H)(|V(G)| - 1).$$

We let $V(G) = \{g_1, g_2, \dots, g_n\}$ and $V(H) = \{h_1, h_2, \dots, h_m\}$. Now suppose C is a dominating set of H . Then we will show that $|S|$ is an upper bound for $\gamma_P(G \square H)$ such that $S = \{(g_i, h) | i \in [n-1], h \in C\}$. Since $N[C] = V(H)$, all the vertices of the form (g_i, h) where $i \in [n-1]$ are dominated by S . Furthermore, the only unobserved vertices in $G \square H$ will be of the form (g_n, h) which are adjacent to vertices labeled (g_{n-1}, h) . Since (g_n, h) is the only unobserved neighbor of (g_{n-1}, h) at timestep 1, it follows that (g_n, h) is observed at timestep 2. Thus, S is in fact a power dominating set. $\square$

To see that the bound in Theorem 8 is sharp, consider $K_3 \square K_{1,3}$, depicted in Figure 2. One can easily verify that $\gamma_P(K_3 \square K_{1,3}) > 1$. If we let S be the set of red vertices, then the green vertices are dominated by S and the blue vertices are observed in timestep 2. Thus, S is in fact a power dominating set of size $2 = \gamma(K_{1,3})(|V(K_3)| - 1)$.

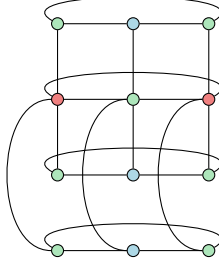


Figure 2: $K_3 \square K_{1,3}$

We can generalize this to all graphs of the form $K_n \square K_{1,m}$ as conjectured in [7].

Theorem 9. For $m \geq n \geq 3$, $\gamma_P(K_n \square K_{1,m}) = n - 1$.

Proof. From Theorem 8, we know that $n - 1$ is an upper bound for $\gamma_p(K_n \square K_{1,m})$ because $\gamma(K_{1,m}) = 1$. Now we will show that $n - 1$ is the least upper bound for $K_n \square K_{1,m}$. For the sake of contradiction, assume there exists a power dominating set S of $K_n \square K_{1,m}$ of cardinality $n - 2$.

Enumerate the vertices of K_n as $V(K_n) = \{x_1, \dots, x_n\}$ and enumerate the vertices of $K_{1,m}$ as $V(K_{1,m}) = \{y_1, \dots, y_{m+1}\}$ where y_1 is the universal vertex. Suppose first that every vertex of S is of the form (x_i, y_1) . Without loss of generality, assume $S = \{(x_i, y_1) \mid 1 \leq i \leq n - 2\}$. Note that

$$N(S) = \{(x_i, y_j) \mid 1 \leq i \leq n - 2, 2 \leq j \leq m + 1\} \cup \{(x_{n-1}, y_1), (x_n, y_1)\}.$$

However, each vertex of $N(S)$ contains at least two unobserved neighbors. Thus, S is not a power dominating set $K_n \square K_{1,m}$. So instead we shall assume that not every vertex of S is of the form (x_i, y_1) . Note that since $|V(K_{1,m})| = m + 1 \geq n + 1 = |S| + 3$, there exist at least 3 G -layers that contain no vertex of S . Reindexing if necessary, we may assume that $G^{y_1}, \dots, G^{y_\alpha}$ each contain a vertex of S and $G^{y_{\alpha+1}}, \dots, G^{y_{m+1}}$ do not contain a vertex of S . Similarly, we may assume $H^{x_1}, \dots, H^{x_\beta}$ each contain a vertex of S and $H^{x_{\beta+1}}, \dots, H^{x_n}$ do not contain a vertex of S . Each vertex of the form (x_i, y_j) for $\beta + 1 \leq i \leq n$ and $1 \leq j \leq \alpha$ is dominated, yet has at least two unobserved neighbors. Similarly, each vertex of the form (x_i, y_j) where $1 \leq i \leq \beta$ and $\alpha + 1 \leq j \leq m + 1$ has at least two unobserved neighbors. Thus, S is not a power dominating set of $K_n \square K_{1,m}$. $\square$

In [4], the following problem was posed.

Problem 1. Study the power domination problem for the graph $G_1 \square G_2 \square G_3$ where $G_i \in \{K_n, P_n, W_n, C_n, K_{1,n}\}$ for $i \in \{1, 2, 3\}$.

We consider several families of such graphs.

Theorem 10. For any $n \geq 3$, $\gamma_P(C_n \square C_n \square C_n) \leq n^2$.

Proof. Suppose $G = C_n \square C_n \square C_n$ and each vertex has the form (i, j, k) where $i, j, k \in [n]$. We claim $S = \{(\lfloor \frac{n}{2} \rfloor + 1, j, k) \mid j, k \in [n]\}$ is a power dominating set of G . Note that the closed neighborhood of S will be $S_1 = \{(i, j, k) \mid i \in \{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor + 1, \lfloor \frac{n}{2} \rfloor + 2\}, j, k \in [n]\}$. So in timestep 1, S_1 will be observed.

We show $A_t = \{(i, j, k) \mid \lfloor \frac{n}{2} \rfloor - t + 1 \leq i \leq \lfloor \frac{n}{2} \rfloor + t + 1, j, k \in [n]\} \subseteq S_t$ where $2 \leq t \leq \lfloor \frac{n}{2} \rfloor$. To see this, let $(i, j, k) \in A_{t+1} - A_t$. Thus, $i \in \{\lfloor \frac{n}{2} \rfloor - t, \lfloor \frac{n}{2} \rfloor + t + 2\}$. So (i, j, k) is the only unobserved neighbor of either $(\lfloor \frac{n}{2} \rfloor - t + 1, j, k)$ or $(\lfloor \frac{n}{2} \rfloor + t + 1, j, k)$. Thus, (i, j, k) is observed in timestep $t + 1$. So all vertices of G will be observed in $\lfloor \frac{n}{2} \rfloor$ timesteps and $\gamma_P(C_n \square C_n \square C_n) \leq n^2$. $\square$

Theorem 11. For $n \geq 1$, $\gamma_P(P_n \square P_n \square P_n) \leq n \times \lceil \frac{n}{3} \rceil$.

Proof. Suppose $H = P_n \square P_n \square P_n$ and each vertex has the form (i, j, k) where $i, j, k \in [n]$. Let D be a dominating set of P_n of cardinality $\lceil \frac{n}{3} \rceil$. We claim $S = \{(i, j, 1) \mid i \in D, j \in [n]\}$ is a power dominating set of H . Note that $S_1 = \{(i, j, 1) \mid i, j \in [n]\} \cup \{(i, j, 2) \mid i \in D, j \in [n]\}$ is the closed neighborhood of S and all vertices of S_1 are observed in timestep 1. We define $A_t = \{(i, j, t) \mid i, j \in [n], t \in [n]\}$. A_{t+1} is in the closed neighborhood of A_t . Note that any vertex of $A_{t+1} - A_t$, say $(i, j, t+1)$, will be the only unobserved neighbor of $(i, j, t) \in A_t$. Thus, each vertex of A_{t+1} is observed by timestep $t+1$. It follows that S is a power dominating set of H and all vertices are observed in n timesteps. $\square$

Theorem 12. For $n \geq 1$, $\gamma_P(K_{1,n} \square K_{1,n} \square K_{1,n}) \leq n(n-1)$.

Proof. Suppose $H = K_{1,n} \square K_{1,n} \square K_{1,n}$ and vertices have the form (i, j, k) where $i, j, k \in [n+1]$ and in each factor $K_{1,n}$, 1 is the vertex with degree n . We show $S = \{(i, j, 1) \mid i \in [n], j \in [n+1]\}$ is a power dominating set of H . Note that $S_1 = \{(i, j, k) \mid i \in [n], j, k \in [n+1]\} \cup \{(n+1, j, 1) \mid j \in [n+1]\}$ is the closed neighborhood of S . So in timestep 1, each vertex of S_1 will be observed.

The only unobserved vertices in H will be in $P = \{(n+1, j, k) \mid j \in [n+1], k \in \{2, 3, \dots, n+1\}\}$. But since the vertices of the form $\{(1, j, k) \mid j, k \in [n+1]\}$ are in S_1 and each has only one unobserved neighbor in P , all the vertices in P will be observed. Notice that $V(H) = S_1 \cup P$. So all vertices of H will be observed in 2 timesteps. $\square$

A wheel denoted by W_n is defined as a graph that contains a cycle of $n-1$ vertices and each of the vertex on the cycle is connected to one other vertex called the central vertex. W_6 is depicted in Figure 3. Note that the following was proven in [4].

Theorem 13. [4] For $n, m \geq 4$, $\gamma_P(W_n \square W_m) \leq 3$.

Theorem 14. For $n \geq 4$, $\gamma_P(W_n \square W_n) = 3$.

Proof. Note that based on Theorem 13, we need only to show that $\gamma_P(W_n \square W_n) \geq 3$. Enumerate the vertices of W_n as $\{x_1, \dots, x_n\}$ such that x_1 is the universal vertex of W_n . Suppose S is a power dominating set of $W_n \square W_n$ of cardinality 2. We know that $\gamma(W_n \square W_n) \geq n$ as $\gamma(G \square H) \geq \min\{|V(G)|, |V(H)|\}$. Therefore, there exists a vertex not dominated by S . Moreover, there exists a vertex (a, b) that is observed in timestep 2, meaning (a, b) is the only unobserved neighbor of some $(w, z) \in S_1$. Suppose first that $z = b$. Since $\deg_G(w) \geq 3$ and $\deg_H(z) \geq 3$, $G^z \cap S \neq \emptyset$ and $H^w \cap S \neq \emptyset$. It follows that $(x_1, z) \in S$ and $(w, x_1) \in S$ for otherwise all horizontal neighbors of (w, z) other than (a, z) are not dominated and all vertical neighbors

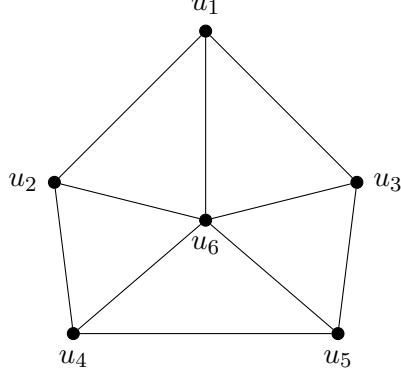


Figure 3: W_6

of (w, z) are not dominated. However, (a, z) is adjacent to (x_1, z) contradicting the assumption that (a, z) is not in S_1 . A similar argument shows S is not a power dominating set if we had instead assumed $w = a$.

□

Theorem 15. *For $n \geq 3$, $\gamma_P(W_n \square W_n \square W_n) \leq 3n$.*

Proof. Since the order of W_n is n , it follows from Theorem 14 and Theorem 8 that $\gamma_P(W_n \square W_n \square W_n) \leq 3n$. □

Next, we turn our attention to a lower bound for $\gamma_P(G \square H)$. Note that the following was shown by Koh and Soh in [4].

Lemma 1. [4] *For any connected graph G and H , $\gamma_P(G \square H) \geq \gamma_P(H)$.*

In [4], it was also claimed that $\gamma_P(G \square T) \geq \gamma_P(G)\gamma_P(T)$ where T is any tree. The proof provided relies on the following. A *spider* is a tree with at most one vertex having degree 3 or more. The *spider number* of a tree T , denoted $sp(T)$, is the minimum number of subsets into which $V(T)$ can be partitioned so that each subset induces a spider. Such a partition is referred to as a *spider partition* and each set of the partition a *spider subset*. The following was proven by Haynes et al. [3].

Theorem 16. [3] *For any tree T , $\gamma_P(T) = sp(T)$.*

Koh and Soh [4] used the above result to show $\gamma_P(G \square T) \geq \gamma_P(G)\gamma_P(T)$. However, there is a hole in their proof, which we provide here in order to address the issue.

Theorem 17. [4] *For any graph G and any tree T , $\gamma_P(G)\gamma_P(T) \leq \gamma_P(G \square T)$.*

Proof. Let S be a power dominating set of $G \square T$ where $m = \gamma_P(G)$ and $n = \gamma_P(T)$. By Theorem 16 [4], we can partition $V(T)$ into n spider subsets $V_1, V_2, \dots, V_n$. Let $T_i = G[V_i]$, the subgraph induced by V_i , where $1 \leq i \leq n$. By Lemma 1, $\gamma_P(G \square T_i) \geq \gamma_P(G) = m$. It follows that $|S \cap V(G \square T_i)| \geq m$, and therefore

$$\gamma_P(G \square T) = |S| = \sum_{i=1}^n |S \cap V(G \square T_i)| \geq mn.$$

□

The issue in the above proof is the conclusion that $\gamma_P(G \square T_i) \geq \gamma_P(G)$ implies that $|S \cap V(G \square T_i)| \geq m$. We provide an example where this need not be the case. Consider the graph $G \square T$ depicted in Figure 4. We claim that $sp(T) = 4$. To see this, suppose there exists a spider partition of size 3, say V_1, V_2 , and V_3 . We may assume that $v_3 \in V_1$, $v_{12} \in V_2$ and $v_{18} \in V_3$. However, this implies that v_7 cannot be in V_1 for otherwise V_1 contains two vertices, namely v_3 and v_5 with degree 3. Similarly, v_8 cannot be in V_2 and v_{15} cannot be in V_1 . Therefore, no such spider partition exists. However, $V_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$, $V_2 = \{v_7, v_8, v_{15}\}$, $V_3 = \{v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$, and $V_4 = \{v_{16}, v_{17}, v_{18}, v_{19}, v_{20}\}$ is a spider partition. Furthermore, $D = V(G) \times \{v_3, v_5, v_9, v_{12}, v_{16}, v_{18}\}$, illustrated by the black vertices, is a power dominating set as it is a dominating set of $G \square T$ yet $D \cap (V(G) \times V_2) = \emptyset$. It is quite possible that the statement of Theorem 17 is correct.

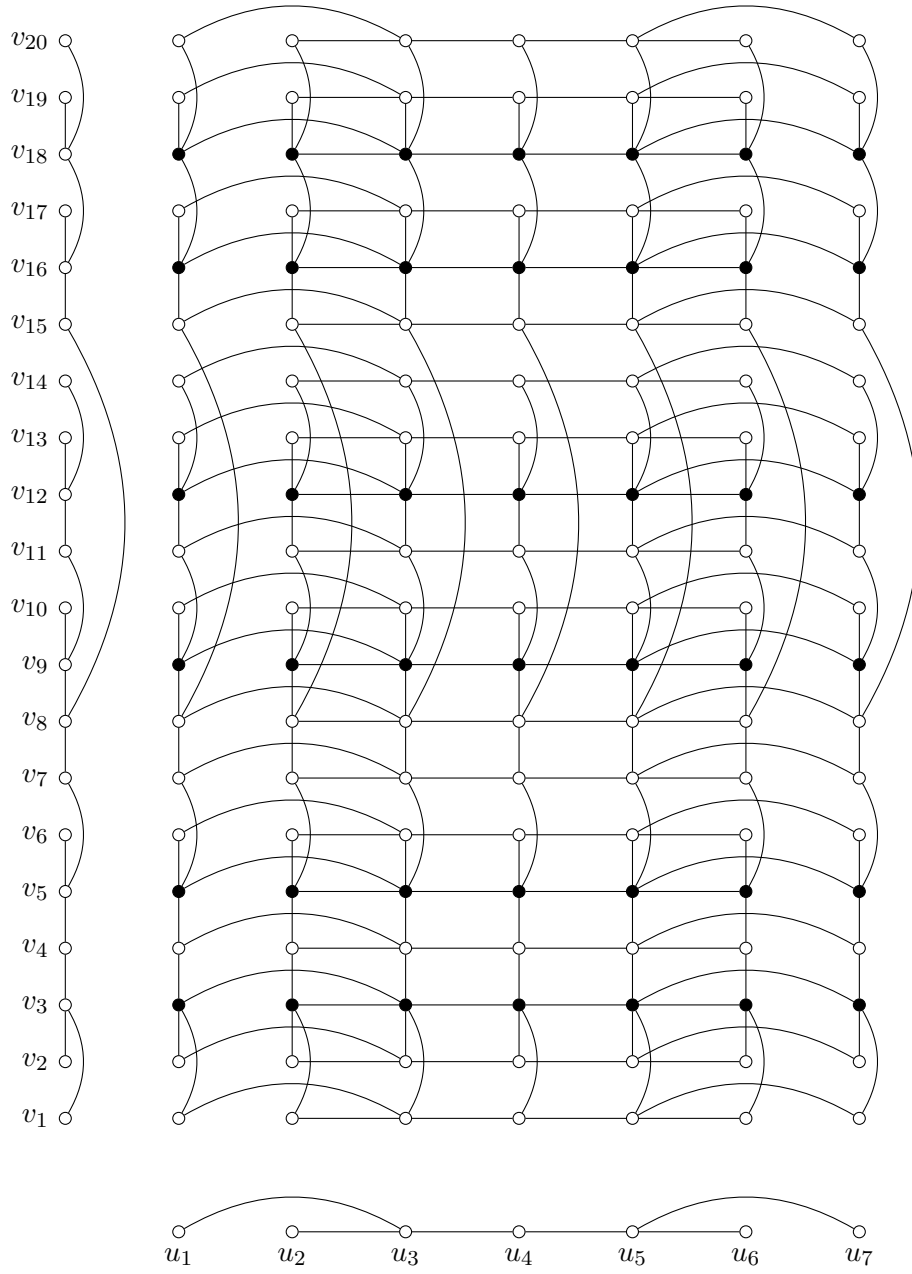


Figure 4: $G \square T$

3 Cubic Graphs

Cubic graphs are graphs all of whose vertices have degree 3 and therefore are 3-regular graphs. The order of a cubic graph is always even. A cubic graph of 8 vertices is shown in Figure 5.

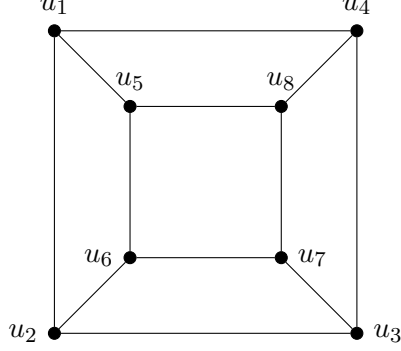


Figure 5: Cubic Graph on 8 vertices

In this section, we will discuss global bounds on cubic graphs and some classes of cubic graphs such as the Circulant graphs.

3.1 Global Upper Bound

In this section we consider power dominating cubic graphs. Note that Zhao et al. [8] were able to show that if G is a connected claw-free cubic graph of order n , then $\gamma_P(G) \leq n/4$. They were also able to classify all extremal graphs where equality holds. To do so, they defined the following class of graphs. The diamond graph is the graph D obtained from K_4 by deleting one edge. For each positive integer k , let D_k be the connected claw-free cubic graph formed from k disjoint copies of D by joining pairwise $2k$ vertices of degree two.

Theorem 18. [8] *If G is a connected claw-free cubic graph of order n , then $\gamma_P(G) \leq n/4$ with equality if and only if $G \in \mathcal{A}$.*

We prove that for any 2-connected cubic graph G of order n , $\gamma_P(G) \leq n/3$. To do this, we utilize the well-known result due to Petersen.

Theorem 19. [6] *Every bridgeless cubic multigraph contains a 2-factor.*

We will consider power dominating sets in cubic multigraphs. Note that if we start with a set $S \subset V(G)$ where G is a multigraph and for some observed vertex v , w is the only unobserved neighbor of v but there are multiple edges between v

and w , then technically, w would not be observed by v . Therefore, when considering multigraphs, special care must be used in the forcing steps in that we can't merely assume that because w is the only unobserved neighbor of v that w will in fact be observed.

Theorem 20. *Suppose G is a 2-connected cubic multigraph of order n . For any vertex $x \in V(G)$, there exists a power dominating set of cardinality at most $n/3$ containing x .*

Proof. Let $\mathcal{C} = \{C_1, \dots, C_\ell\}$ be a 2-factor of G and write $C_i = v_{i,1}v_{i,2} \dots v_{i,n_i}v_{i,1}$ where n_i is the order of C_i . Notice that if there are multiple edges between u and v in G , then u and v must be on the same cycle C_i . Reindexing if necessary, we may assume $C_1, \dots, C_k$ are those cycles of $\mathcal{C}$ where $n_i \equiv 1, 2 \pmod{3}$. We create a graph H as follows. Let $V(H) = \{u_1, \dots, u_\ell\}$ where $u_i u_{i'} \in E(H)$ if there exists a vertex on C_i that is adjacent to a vertex on $C_{i'}$ where $1 \leq i < i' \leq \ell$. Choose a spanning tree T of H and choose any vertex $u_r \in V(H)$. We root T at u_r and define $A_i = \{u_j \mid d(u_r, u_j) = i\}$ for $1 \leq i \leq d$ where $d = \max_{u_i} \{d(u_r, u_i)\}$. We recursively reindex the vertices of C_i for $1 \leq i \leq \ell$ as follows. First, choose any $u_i \in A_1$. Choose a representative edge $v_{r,j}v_{i,j'}$ in G between C_r and C_i . Reindex the vertices of C_r and C_i such that $v_{r,j} = v_{r,1}$ and $v_{i,j'} = v_{i,1}$. Now suppose we have already reindexed the vertices of C_t where $u_t \in A_i$ for $1 \leq i \leq \beta - 1$ for some $\beta \in \mathbb{N}$. Let $u_s \in A_\beta$. Thus, there exists a unique edge in T of the form $u_s u_t$ where $u_t \in A_{\beta-1}$. Choose a representative edge $v_{s,j}v_{t,j'}$ in G between C_s and C_t . Reindex the vertices of C_s such that $v_{s,j} = v_{s,1}$.

Suppose x is on C_t where $1 \leq t \leq \ell$ and write $x = v_{t,y}$. Note that we may assume that if $y \equiv 1 \pmod{3}$, then $1 \leq y \leq n_t - 1$ as we can always choose to index clockwise or counter-clockwise. Similarly, if $y \equiv 2 \pmod{3}$, we can assume $1 \leq y \leq n_t - 2$. We proceed by considering the following cases based on the value of k .

Case 1 $k = 1$ and $\ell \neq k$

Suppose first that x is on C_1 . Let $S_1 = \{v_{1,a} \mid a \equiv y \pmod{3}, 1 \leq a \leq n_1\}$. Reindexing $C_2, \dots, C_\ell$ if need be, we may assume u_2 is adjacent to u_1 in T . Thus, $v_{2,1}$ is adjacent to a vertex on C_1 . If $n_2 = 3$, then set $S_2 = \emptyset$. Otherwise, set $S_2 = \{v_{2,a} \mid a \equiv 0 \pmod{3}, 1 \leq a < n_2\}$. For $3 \leq i \leq \ell$, set $S_i = \{v_{i,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_i\}$. We claim that $S = \cup_i S_i$ is a power dominating set of G that contains x . Note that S dominates all vertices of G except perhaps $v_{2,1}, v_{2,n_2}$, and v_{2,n_2-1} . If $n_2 > 3$, then $v_{2,1}$ is dominated by S_1 or is the only unobserved neighbor of some vertex on C_1 and is therefore observed in timestep 2. Since there is only one edge between $v_{2,1}$ and v_{2,n_2} , v_{2,n_2} will be observed by timestep 3. Finally, if there are two edges between v_{2,n_2} and v_{2,n_2-1} , then there is only one edge between v_{2,n_2-1} and v_{2,n_2-2} and v_{2,n_2-1} will be observed. On the other hand, if there is only one

edge between v_{2,n_2} and v_{2,n_2-1} , then v_{2,n_2-1} will be observed by timestep 3. Thus, G is in fact a power dominating set. If $n_2 = 3$, then both $v_{2,2}$ and $v_{2,3}$ are adjacent to a vertex on another cycle for otherwise the edge between $v_{2,1}$ and C_1 is a bridge. Thus, $v_{2,2}$ (and $v_{2,3}$) will be the only unobserved neighbor of some vertex on C_i for $i \neq 2$. It follows that G is a power dominating set. Moreover,

$$|S| = \left\lceil \frac{n_1}{3} \right\rceil + \frac{n_2}{3} - 1 + \sum_{i=3}^{\ell} \frac{n_i}{3} \leq \frac{n}{3}.$$

Finally, suppose x is on C_t for $2 \leq t \leq \ell$. In this case, we root T at u_t and let $S_i = \{v_{i,a} \mid a \equiv y \pmod{3}, 1 \leq a \leq n_i\}$ for $2 \leq i \leq \ell$. We let $S_1 = \{v_{1,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_1 - 1\}$. Note that $S = \cup_i S_i$ dominates all vertices of G other than perhaps $v_{1,1}$ and v_{1,n_1} . Since $v_{1,1}$ is adjacent to some vertex w on cycle C_i where $2 \leq i \leq \ell$, $v_{1,1}$ will be the only unobserved neighbor of w and therefore observed by timestep 2. Furthermore, there is only one edge between $v_{1,1}$ and v_{1,n_1} so v_{1,n_1} will be observed by timestep 3. It follows that S is a power dominating set of G of cardinality

$$|S| = \left\lfloor \frac{n_1}{3} \right\rfloor + \sum_{i=2}^{\ell} \frac{n_i}{3} < \frac{n}{3}.$$

Case 2 $k = \ell = 1$

In this case, G is Hamiltonian and $n \geq 4$. If $n = 4$, then $G = K_4$ and $\gamma_P(G) = 1$. So we shall assume $n > 4$. If $n \equiv 0 \pmod{3}$, then $\gamma_P(G) \leq \gamma(G) \leq n/3$ and we can choose a power dominating set of cardinality $n/3$ containing x . So we shall assume $n \equiv 1, 2 \pmod{3}$. In this case, we enumerate the vertices of C as $C = v_1 v_2 \dots v_n v_1$ such that $x = v_3$ and we have indexed in the clockwise direction. If $n \equiv 1 \pmod{3}$, then $S = \{v_i \mid i \equiv 0 \pmod{3}, 1 \leq i \leq n - 1\}$ dominates all vertices except v_1 . It follows that there is only one edge between v_1 and v_2 or only one edge between v_1 and v_n . In either case, v_1 will be observed by timestep 2. So we shall consider when $n \equiv 2 \pmod{3}$. Consider S defined above and notice that potentially the only vertices not dominated by S are v_1 and v_n . If there is only one edge between v_1 and v_2 , then v_1 will be observed by timestep 2 provided $v_2 v_n \notin E(G)$. Furthermore, this implies there is only one edge between v_1 and v_n or only one edge between v_n and v_{n-1} . In either case, v_n will also be observed. On the other hand, if $v_2 v_n \in E(G)$, then there is only one edge between v_{n-1} and v_n so v_n is observed by timestep 2 and v_1 is observed by timestep 3. So we shall assume that there are two edges between v_1 and v_2 . We shall also assume that there are two edges between v_4 and v_5 for S would be a power dominating set of G otherwise if we had originally indexed in the counter-clockwise direction. This immediately implies $n > 5$. Note that if there is only one edge between v_n and v_{n-1} , then S is still a power dominating set of G as v_n is the only unobserved neighbor of v_{n-1} and since there is only one edge

between v_1 and v_n , v_1 is the only unobserved neighbor of v_n . So we shall assume there are two edges between v_n and v_{n-1} . Moreover, if $n = 8$, then $v_3v_6 \in E(G)$ and $S = \{v_3, v_7\}$ is a power dominating set of G . So we shall assume $n > 8$ and there are two edges between v_6 and v_7 . Note that x must be adjacent to some v_j where $j \notin \{1, 2, n, n-1, 4, 5, 6, 7\}$. Suppose first that $j \equiv 0 \pmod{3}$. Define $S' = \{v_i \mid i \equiv 0 \pmod{3}, 3 \leq i < j\} \cup \{v_i \mid i \equiv 1 \pmod{3}, j+1 \leq i \leq n-1\}$. Note that the only vertices that are potentially not dominated by S' are v_{j-1} and v_1 . However, v_n is dominated and there is only one edge between v_n and v_1 so v_1 will be observed. Similarly, there is only one edge between v_j and v_{j-1} and v_j is dominated. Thus, v_{j-1} will be observed. Thus, S' is a power dominating set where $|S'| = \lfloor n/3 \rfloor$. One can easily verify that if $j \equiv 1 \pmod{3}$, then $S' = \{v_i \mid i \equiv 0 \pmod{3}, 3 \leq i \leq j-1\} \cup \{v_i \mid i \equiv 1 \pmod{3}, j+2 \leq i \leq n-1\}$ is a power dominating set of G and if $j \equiv 2 \pmod{3}$, then $S' = \{v_i \mid i \equiv 0 \pmod{3}, 3 \leq i \leq j-2\} \cup \{v_i \mid i \equiv 1 \pmod{3}, j+1 \leq i \leq n-1\}$ is a power dominating set of G .

Case 3 $k = \ell = 2$

We shall assume that x is on C_1 and root T at u_1 . Note that there exist at least two edges between C_1 and C_2 for otherwise G contains a bridge. Thus, we choose representative edge $v_{1,1}v_{2,1}$ such that $v_{1,1} \neq x$. Again, we may assume $x = v_{1,y}$ where $2 \leq y \leq n_1 - 1$. Suppose first that $n_1 \equiv 2 \pmod{3}$ or $n_2 \equiv 2 \pmod{3}$. If $y \equiv 1, 2 \pmod{3}$, consider $S = \{v_{1,a} \mid a \equiv y \pmod{3}, 1 \leq a \leq n_1\} \cup \{v_{2,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_2\}$. Otherwise, let $S = \{v_{1,a} \mid a \equiv y \pmod{3}, 1 \leq a \leq n_1\} \cup \{v_{1,n_1}\} \cup \{v_{2,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_2\}$. Note that the only vertices that are perhaps not dominated by S are $v_{2,1}$ and v_{2,n_2} . Note that $v_{2,1}$ is the only unobserved neighbor of $v_{1,1}$ so it is observed by timestep 2. Moreover, v_{2,n_2} is the only unobserved neighbor of $v_{2,1}$ so it is observed by timestep 3. Thus, S is a power dominating set of cardinality $|S| = \lceil n_1/3 \rceil + \lfloor n_2/3 \rfloor \leq n/3$. So we shall assume $n_1 \equiv 1 \pmod{3}$ and $n_2 \equiv 1 \pmod{3}$.

Suppose first that $y \equiv 0 \pmod{3}$. Consider $S = \{v_{1,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_1\} \cup \{v_{2,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_2\}$. The only vertices perhaps not dominated by S are $v_{1,1}$ and $v_{2,1}$. In this case, $v_{1,1}$ is the only unobserved neighbor of $v_{1,2}$ and $v_{2,1}$ is the only unobserved neighbor of $v_{2,2}$. So S is indeed a power dominating set containing x of cardinality $\lfloor n_1/3 \rfloor + \lfloor n_2/3 \rfloor$.

Next, suppose $y \equiv 1 \pmod{3}$ and consider $S = \{v_{1,a} \mid a \equiv 1 \pmod{3}, 4 \leq a \leq n_1\} \cup \{v_{2,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_2\}$. Note that the only vertices potentially not dominated by S are $v_{1,2}$ and $v_{2,1}$. We may assume that either $v_{2,2}$ is not adjacent to $v_{1,2}$ or v_{2,n_2} is not adjacent to $v_{1,2}$. In either case, $v_{2,1}$ is the only unobserved neighbor of $v_{2,2}$ or v_{2,n_2} and there is only one edge between $v_{2,1}$ and its neighbors on C_2 . Thus, $v_{2,1}$ will be observed by timestep 2. Thus, $v_{1,2}$ will be observed by timestep 3 as it is the only unobserved neighbor of $v_{1,1}$. So S is a power dominating set containing x of cardinality $\lfloor n_1/3 \rfloor + \lfloor n_2/3 \rfloor$.

Finally, suppose $y \equiv 2 \pmod{3}$ and consider $S = \{v_{1,a} \mid a \equiv 2 \pmod{3}, 2 \leq a \leq n_1\} \cup \{v_{1,n_1}\} \cup \{v_{2,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_2\}$. Note that the only vertices that are perhaps not dominated by S are $v_{1,2}$ and $v_{2,1}$. Using a similar argument as that used in the case that $y \equiv 1 \pmod{3}$, one can easily verify that S is a power dominating set containing x of cardinality $\lfloor n_1/3 \rfloor + \lfloor n_2/3 \rfloor$.

Case 4 $k = 2$ and $\ell \neq k$

Suppose first that x is on C_1 or C_2 . Without loss of generality, we may assume x is on C_1 and we root T at u_1 . If there exists a spanning tree T of H where $u_1 u_2 \in E(T)$, then based on Case 3, we can choose a set of vertices from $C_1 \cup C_2$ of cardinality $\lfloor (n_1 + n_2)/3 \rfloor$, call it A , such that $S = A \cup \bigcup_{i=3}^{\ell} \{v_{i,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_i\}$ will be a power dominating set of G containing x of cardinality no larger than $n/3$. So we shall assume no such spanning tree exists and that $u_2 \in A_s$ for some $2 \leq s \leq d$.

Suppose first that $n_1 \equiv 2 \pmod{3}$ or $n_2 \equiv 2 \pmod{3}$. If $y \equiv 0 \pmod{3}$, then let $S_1 = \{v_{1,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_1\} \cup \{v_{1,1}\}$. If $y \equiv 1, 2 \pmod{3}$, let $S_1 = \{v_{1,a} \mid a \equiv y \pmod{3}, 1 \leq a \leq n_1\}$. Note that in each situation, we have chosen a set of cardinality $\lceil n_1/3 \rceil$. For $2 \leq i \leq \ell$, let $S_i = \{v_{i,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_i\}$ and consider $S = \bigcup_i S_i$. Note that the only vertices that are perhaps not dominated by S are $v_{2,1}$ and v_{2,n_2} . Since $v_{2,1}$ is adjacent to some w on C_i for $3 \leq i \leq \ell$, it follows that $v_{2,1}$ is the only unobserved neighbor of w and therefore observed by timestep 2. Furthermore, there is only one edge between $v_{2,1}$ and v_{2,n_2} so v_{2,n_2} will be observed by timestep 3. It follows that S is a power dominating set containing x of cardinality

$$|S| = \left\lceil \frac{n_1}{3} \right\rceil + \sum_{i=2}^{\ell} \left\lfloor \frac{n_i}{3} \right\rfloor \leq \frac{n}{3}.$$

So we shall assume $n_1 \equiv 1 \pmod{3}$ and $n_2 \equiv 1 \pmod{3}$. As above, if $y \equiv 0 \pmod{3}$, then let $S_1 = \{v_{1,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_1\} \cup \{v_{1,1}\}$. If $y \equiv 1, 2 \pmod{3}$, let $S_1 = \{v_{1,a} \mid a \equiv y \pmod{3}, 1 \leq a \leq n_1\}$. Note that in each situation, we have chosen a set of cardinality $\lceil n_1/3 \rceil$. Furthermore, reindexing the cycles $C_3, \dots, C_{\ell}$, we may assume that u_3 is adjacent to u_1 in T . We shall assume first that $n_3 > 3$. We let $S_3 = \{v_{3,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_3 - 3\}$ and for each $i \notin \{1, 3\}$, we let $S_i = \{v_{i,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_i\}$. Note that the only vertices that $S = \bigcup_i S_i$ perhaps does not dominate are $v_{3,1}, v_{3,n_3}, v_{3,n_3-1}$ and $v_{2,1}$. Note that $v_{3,1}$ is the only unobserved neighbor of some vertex on C_1 so it is observed by timestep 2. Similarly, v_{3,n_3} is the only unobserved neighbor of $v_{3,1}$ in timestep 2 and there is only one edge between $v_{3,1}$ and v_{3,n_3} . So v_{3,n_3} will also be observed. Next, note that either $v_{2,2}$ or v_{2,n_2} will not be adjacent to v_{3,n_3-1} so $v_{2,1}$ is the only unobserved neighbor of one of these two vertices and $v_{2,1}$ is observed. It follows that v_{3,n_3-1} will be the only unobserved neighbor sharing only one edge with either v_{3,n_3} or v_{3,n_3-2} . Thus, S is in fact a power dominating set containing x of cardinality at

most $n/3$. So now assume $n_3 = 3$ and let $S_3 = \emptyset$ and for each $i \notin \{1, 3\}$, we let $S_i = \{v_{i,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_i\}$. In this case, we know that there is only one edge between $v_{3,2}$ and $v_{3,3}$ for otherwise G contains a bridge. Note that the only vertices that $S = \cup_i S_i$ perhaps does not dominate are $v_{3,1}, v_{3,2}, v_{3,3}$ and $v_{2,1}$. As above, $v_{3,1}$ is the only unobserved neighbor of some vertex on C_1 so it is observed by timestep 2. Furthermore, if $v_{2,1}$ is not adjacent to either $v_{3,2}$ or $v_{3,3}$, then each of $v_{2,1}, v_{3,2}$ and $v_{3,3}$ will be the only unobserved neighbor of some vertex not on their cycle. So we shall assume without loss of generality that $v_{2,1}$ is adjacent to $v_{3,2}$. It follows that $v_{3,3}$ is the only unobserved neighbor of some vertex not on C_3 and is therefore observed by timestep 2. Thus, $v_{3,2}$ is observed by timestep 3 and $v_{2,1}$ is observed by timestep 4. Thus, S is a power dominating set containing x of cardinality at most $n/3$.

Finally, we assume that x is on C_i for $3 \leq i \leq \ell$. Without loss of generality, we may assume x is on C_3 and we root T at u_3 . For all $3 \leq i \leq \ell$, let $S_i = \{v_{i,a} \mid a \equiv y \pmod{3}, 1 \leq a \leq n_i\}$ and for $i \in \{1, 2\}$, let $S_i = \{v_{i,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_i\}$. Note that the only vertices that are potentially not dominated by $S = \cup_i S_i$ are $v_{1,1}, v_{1,n_1}, v_{2,1}$, and v_{2,n_2} . If $u_1 u_2 \in E(T)$, then we may assume $d(u_3, u_1) < d(u_3, u_2)$. Suppose $u_1 \in A_s$ for some $2 \leq s \leq d$. It follows that $v_{1,1}$ is the only unobserved neighbor of some vertex on C_j where $u_j \in A_{s-1}$ and $3 \leq j \leq \ell$. Thus, $v_{1,1}$ will be observed by timestep 2. Moreover, v_{1,n_1} shares only one edge with $v_{1,1}$ so it too will be observed by timestep 3. Similarly, $v_{2,1}$ will be the only unobserved neighbor of some vertex on C_j for $j \neq 2$ in timestep 3 so it will be observed by timestep 4. It follows that v_{2,n_2} will also be observed. Thus, S is a power dominating set containing x of cardinality at most $n/3$.

Case 5 $k \geq 3$

Suppose first that x is on C_i for $1 \leq i \leq k$. Without loss of generality, we shall assume x is on C_1 and we root T at u_1 . If $y \equiv 0 \pmod{3}$, then let $S_1 = \{v_{1,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_1\} \cup \{v_{1,1}\}$. If $y \equiv 1, 2 \pmod{3}$, let $S_1 = \{v_{1,a} \mid a \equiv y \pmod{3}, 1 \leq a \leq n_1\}$. Note that in each situation, we have chosen a set of cardinality $\lceil n_1/3 \rceil$. For $2 \leq i \leq \ell$, let $S_i = \{v_{i,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_i\}$ and consider $S = \cup_i S_i$. Note that the only vertices that are potentially not dominated by S are $v_{i,1}$ and v_{i,n_i} for each $2 \leq i \leq k$.

We show that if $u_i \in A_s$ for some $s \in [d]$, then all vertices of C_i are observed by timestep $2s + 1$. Suppose first that $u_i \in A_1$. Since $v_{i,1}$ is the only unobserved neighbor of some vertex on C_1 , $v_{i,1}$ is observed in timestep 2. Furthermore, if v_{i,n_i} is not observed by timestep 2, then it is observed in timestep 3 as it is the only unobserved neighbor of $v_{i,1}$ and there is only one edge between $v_{i,1}$ and v_{i,n_i} . Therefore, we may assume that all vertices of C_i where $u_i \in A_1$ are observed. So assume that all vertices of C_i where $u_i \in A_s$ for $1 \leq s \leq z - 1$ are observed by timestep $2s + 1$ and let $u_i \in A_z$. Potentially the only vertices on C_i that are

not dominated by S are $v_{i,1}$ and v_{i,n_i} . As we specifically reindexed so that $v_{i,1}$ is adjacent to a vertex on $C_{i'}$ where $u_{i'} \in A_{z-1}$, it follows that $v_{i,1}$ is observed by timestep $2(z-1) + 2 = 2z$. Furthermore, if v_{i,n_i} is not observed by timestep $2z$, then it is observed in timestep $2z+1$ as it is the only unobserved neighbor of $v_{i,1}$. Thus, all vertices of C_i for $1 \leq i \leq k$ will eventually be observed. It follows that S is a power dominating set of G containing x of cardinality

$$|S| \leq \left\lceil \frac{n_1}{3} \right\rceil + \sum_{i=2}^{\ell} \left\lfloor \frac{n_i}{3} \right\rfloor \leq \frac{n_1}{3} + \frac{2}{3} + \left(\sum_{i=2}^{\ell} \frac{n_i}{3} \right) - \frac{2}{3} = \frac{n}{3}$$

Finally, suppose x is on C_t where $k+1 \leq t \leq \ell$ and root T at u_t . In this case, we set $S_t = \{v_{t,a} \mid a \equiv y \pmod{3}, 1 \leq a \leq n_t\}$ and for each $i \neq t$, we set $S_i = \{v_{i,a} \mid a \equiv 0 \pmod{3}, 1 \leq a \leq n_i\}$. Based on the above argument, one can easily verify that $S = \cup_i S_i$ is a power dominating set containing x of cardinality no more than $n/3$. $\square$

3.2 Cubic Graphs with $\gamma_P = 1$

In this section, we show that if G is cubic with $\gamma_P(G) = 1$, then G must contain a triangle. Recall that the girth of a graph G is the length of its smallest cycle and is denoted $g(G)$.

Lemma 2. *A cubic graph G with $g(G) > 3$ will have $\gamma_P(G) > 1$.*

Proof. Suppose there exists a cubic graph G with $g(G) > 3$ and $\gamma_P(G) = 1$. Let $S = \{w\}$ be a power dominating set of G and write $N(w) = \{x, y, z\}$. Note that $G \neq K_4$ as $g(G) > 3$. So there exists a vertex $t \in V(G) - N[w]$ that must be observed at timestep 2. Without loss of generality, we may assume t is the only unobserved neighbor of x at timestep 1. It follows that x must be adjacent to either y or z , implying G contains a triangle. Hence, we have a contradiction and no such graph G exists. $\square$

3.3 Circulant Graphs

Given integer $n \geq 3$ and a list $\ell \subseteq [n]$, we define the circulant graph $Ci_n(\ell)$ as the graph having vertex set $V(Ci_n(\ell)) = \{u_1, \dots, u_n\}$ such that for $1 \leq i \leq n$, u_i is adjacent to u_{i+j} and u_{i-j} for each $j \in \ell$. $Ci_n(\ell) = K_n$ if $\ell = \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$ and $Ci_n(\ell) = C_n$ if $\ell = \{1\}$. Note that circulant graphs are important as they naturally arise in group theory. A more algebraic definition of a circulant graph is the following. A circulant graph is the graph $G(n, S)$ where the vertices of $G(n, S)$ are the elements of the cyclic group Z_n and its edges are those unordered pairs $\{x, y\}$ such that $x - y \in S$ where $0 \notin S \subseteq Z_n$ is a subset of nonzero elements which is

closed under additive inverses. A circulant graph, Ci_n , is a graph of n vertices such that $v_i \in V(Ci_n)$ is adjacent to $v_{(i+j)}$ and $v_{(i-j)}$ vertices for each j in a list l . The circulant graph is denoted by $Ci_n(l)$. $Ci_n(1, 2, \dots, \lfloor \frac{n}{2} \rfloor)$ gives the complete graph K_n and the graph $Ci_n(1)$ gives the n cycle graph C_n .

Theorem 21. *For even integer $n > 4$ and $\ell = \{1, \frac{n}{2}\}$, $\gamma_P(Ci_n(\ell)) = 2$.*

Proof. Suppose $G = V(Ci_n(\ell))$ where $V(G) = \{u_1, \dots, u_n\}$ such that u_i is adjacent to u_{i+1}, u_{i-1} and $u_{i+\frac{n}{2}}$ for $1 \leq i \leq \frac{n}{2}$. Since for $n > 4$, $g(G) > 3$, $\gamma_P(G) > 1$ by Lemma 2. We show that $S = \{u_1, u_4\}$ is a power dominating set of G . Notice $N[S] = \{u_1, u_2, u_3, u_4, u_5, u_n, u_{1+\frac{n}{2}}, u_{4+\frac{n}{2}}\}$. The only unobserved neighbors of u_2 and u_3 are $u_{2+\frac{n}{2}}$ and $u_{3+\frac{n}{2}}$, respectively, which will be observed in timestep 2. Next, $u_{\frac{n}{2}}$ and $u_{5+\frac{n}{2}}$ will be observed in timestep 3 as they are the only unobserved neighbors of $u_{1+\frac{n}{2}}$ and $u_{4+\frac{n}{2}}$, respectively, in timestep 2. Moreover, in timestep $i-1$ for $4 \leq i \leq \frac{n}{2}-3$, u_{i+2} is the only unobserved neighbor of u_{i+1} , and $u_{i+2+\frac{n}{2}}$ is the only unobserved neighbor of $u_{i+1+\frac{n}{2}}$. Thus, $\{u_{i+2}, u_{i+2+\frac{n}{2}}\} \subset S_i$. It follows that S is power dominating set of G . $\square$

Theorem 22. *For even integer $n > 4$, $\ell = \{k, \frac{n}{2}\}$, $k \not\equiv 0 \pmod{2}$ and $n \not\equiv 0 \pmod{k}$, $\gamma_p(Ci_n(\ell)) = 2$.*

Proof. Suppose $G = V(Ci_n(\ell))$ where $V(G) = \{u_1, \dots, u_n\}$ such that u_i is adjacent to u_{i+k}, u_{i-k} and $u_{i+\frac{n}{2}}$ for $1 \leq i, k \leq \frac{n}{2}$. Since for $n > 4$, $g(G) > 4$, $\gamma_p(G) > 1$ by Lemma 2. We can re-index the vertices of G in G' such that $u_1 = v_1$ and $u_{1+ik} = v_i$ for $1 \leq i \leq n-1$. Then G' will have an n -cycle $T = v_1, v_2, \dots, v_n, v_1$ and v_i will be adjacent to $v_{i+1}, v_{i-1}, v_{i+\frac{n}{2}}$. So we can use Theorem 21, as $G' = Ci_n(\{1, \frac{n}{2}\})$ and $\gamma_p(G') = 2$. Furthermore, since $G' \cong G$, $\gamma_p(G) = 2$. $\square$

Theorem 23. *For even integer $n > 4$, $\ell = \{k, \frac{n}{2}\}$, $k \equiv 0 \pmod{2}$ and $n \equiv 0 \pmod{4}$, $\gamma_p(Ci_n(\ell)) = 4$.*

Proof. Suppose $G = V(Ci_n(\ell))$ where $V(G) = \{u_1, \dots, u_n\}$ such that u_i is adjacent to u_{i+k}, u_{i-k} and $u_{i+\frac{n}{2}}$ for $1 \leq i, k \leq \frac{n}{2}$. Since for $n > 4$, $g(G) > 4$, $\gamma_p(G) > 1$ by Lemma 2. We can re-index the vertices of G in A such that $u_{1+ik} = v_i$ for $0 \leq i \leq \frac{n}{2}-1$. Similarly, we can re-index the vertices of G in B such that $u_{\frac{n}{2}+1+ik} = w_i$ for $0 \leq i \leq \frac{n}{2}-1$. Then the graph $G' = A \cup B$ will contain two components A and B . Both components A and B will have $\frac{n}{2}$ -cycles $S = v_1, v_2, \dots, v_{\frac{n}{2}}, v_1$ and $T = w_1, w_2, \dots, w_{\frac{n}{2}}, w_1$ respectively such that v_i will be adjacent to $v_{i+1}, v_{i-1}, v_{i+\frac{n}{2}}$ and w_i will be adjacent to $w_{i+1}, w_{i-1}, w_{i+\frac{n}{2}}$. So we can use Theorem 21, as $A = B = Ci_{\frac{n}{2}}(\{1, \frac{n}{4}\})$ and $\gamma_p(A) = \gamma_p(B) = 2$. Furthermore, since $G' \cong G$, $\gamma_p(G) = \gamma_p(A) + \gamma_p(B) = 4$. $\square$

Theorem 24. *For even integer $n > 4$, $\ell = \{k, \frac{n}{2}\}$, $k \equiv 0 \pmod{2}$ and $n \not\equiv 0 \pmod{4}$ or $n \equiv 0 \pmod{k}$, $\gamma_p(Ci_n(\ell)) = 2$.*

Proof. Suppose $G = V(Ci_n(\ell))$ where $V(G) = \{u_1, \dots, u_n\}$ such that u_i is adjacent to u_{i+1}, u_{i-1} and $u_{i+\frac{n}{2}}$ for $1 \leq i \leq \frac{n}{2}$. Since for $n > 4, g(G) > 4, \gamma_p(G) > 1$ by Lemma 2. by Lemma 2. We show that $S = \{u_1, u_{1+3k}\}$ is a power dominating set of G . Notice $N[S] = \{u_1, u_{1+k}, u_{1+2k}, u_{1+3k}, u_{1+4k}, u_{1+n-k}, u_{1+\frac{n}{2}}, u_{1+3k+\frac{n}{2}}\}$. The only unobserved neighbors of u_{1+k} and u_{1+2k} are $u_{1+k+\frac{n}{2}}$ and $u_{1+2k+\frac{n}{2}}$, respectively, which will be observed in timestep 2. Next, $u_{1+4k+\frac{n}{2}}$ and $u_{1+n-k+\frac{n}{2}}$ will be observed in timestep 3 as they are the only unobserved neighbour of $u_{1+3k+\frac{n}{2}}$ and $u_{1+\frac{n}{2}}$ respectively. Moreover, in timestep i for $5 \leq i \leq \frac{n}{4} - 3$, u_{1+ik} is the only unobserved neighbor of $u_{1+(i-1)k}$, $u_{1+ik+\frac{n}{2}}$ is the only unobserved neighbor of $u_{1+(i-1)k+\frac{n}{2}}$, $u_{1+n-(i-3)k}$ is the only unobserved neighbor of $u_{1+n-(i-4)k}$ and $u_{1+n-(i-3)k+\frac{n}{2}}$ is the only unobserved neighbor of $u_{1+n-(i-4)k+\frac{n}{2}}$. Thus, $\{u_{1+ik}, u_{1+ik+\frac{n}{2}}, u_{1+n-(i-3)k}, u_{1+n-(i-3)k+\frac{n}{2}}\} \subset S_i$. It follows that S is power dominating set of G . $\square$

3.4 Specific Cubic Graphs

LCF notation is used for the representation of cubic graphs that contain a Hamiltonian cycle. The Hamiltonian cycle itself includes two out of the three adjacencies for each vertex. The third edge incident to each vertex is then described as the number of positions clockwise (positive) or counter-clockwise (negative) the third neighbor lies. The basic form of the LCF notation is just the sequence of these numbers of positions, starting from an arbitrarily chosen vertex and written in square brackets. The numbers between the brackets are interpreted modulo N , where N is the number of vertices. Often the pattern repeats, and the number of repetitions can be indicated by a superscript in the notation. For example, the Franklin graph with a corresponding sequence of the position of the third neighbor is shown in Figure 6.

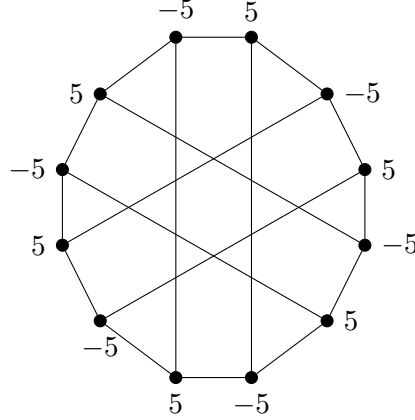


Figure 6: Franklin Graph with LCF notation

The LCF notation for a few graphs are given below:

Graph	Number of Vertices	LCF Notation
Franklin Graph	12	$[5, -5]^6$
Haewood Graph	14	$[5, -5]^7$
Möbius–Kantor Graph	16	$[5, -5]^8$
Zero-symmetric Graph	18	$[5, -5]^9$

All of these graphs are cubic, symmetric, triangle-free, bipartite and Hamiltonian.

Theorem 25. *If G is the Franklin graph, then $\gamma_P(G) = 2$.*

Proof. Since $g(G) \geq 4$, $\gamma_P(G) > 1$ by Lemma 2. We show that $S = \{v_1, v_9\}$, depicted in Figure 7 by the red vertices, is a power dominating set of G . It is clear that $S_1 = N[S] = \{v_1, v_2, v_3, v_6, v_7, v_{10}, v_{12}\}$. Therefore, all vertices of S_1 are colored either green or red in Figure 7. The only unobserved neighbors of v_2 , v_3 and v_6 are v_8 , v_4 and v_5 , respectively, which will be observed in timestep 2. Next, we will observe v_{11} in timestep 3 as it is the only unobserved vertex in G in timestep 2. $\square$

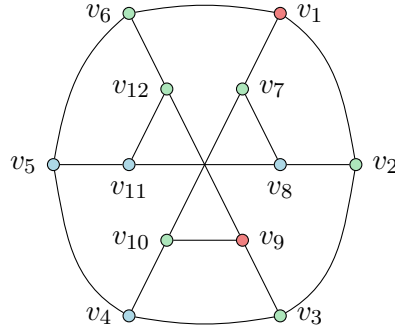


Figure 7: Power dominating set of the Franklin Graph

Theorem 26. *If G is the Heawood graph, then $\gamma_P(G) = 2$.*

Proof. Since $g(G) \geq 4$, $\gamma_P(G) > 1$ by Lemma 2. We show that $S = \{v_1, v_{12}\}$, depicted by the red vertices in Figure 8, is a power dominating set of G . It is clear that $S_1 = N[S] = \{v_1, v_2, v_3, v_{10}, v_{11}, v_{12}, v_{13}, v_{14}\}$. Therefore, all vertices of S_1 are colored red or green in Figure 8. The only unobserved neighbors of v_2 , v_{11} , v_{13} , v_{14} , v_{10} and v_3 are v_7 , v_6 , v_8 , v_5 , v_9 and v_4 , respectively, which will be observed in timestep 2. $\square$

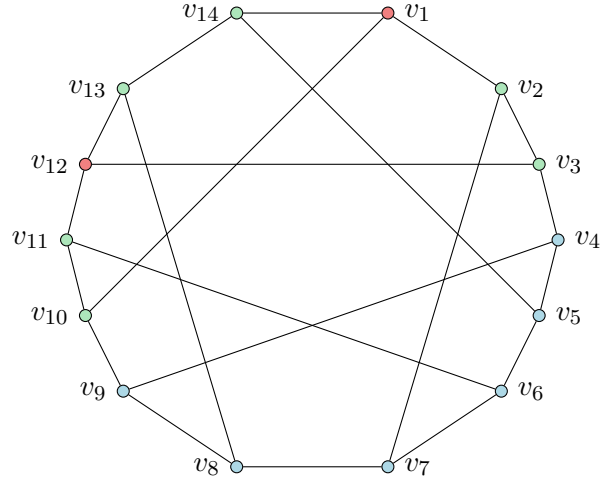


Figure 8: Haewood Graph

Theorem 27. *If G is the Möbius–Kantor graph then $\gamma_P(G) = 2$.*

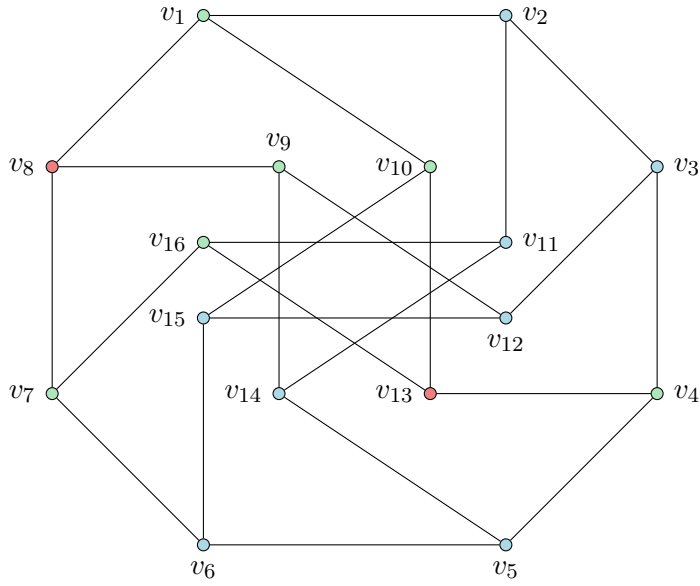


Figure 9: Möbius–Kantor Graph

Proof. Since $g(G) > 4$, $\gamma_P(G) > 1$ by Lemma 2. We show that $S = \{v_8, v_{13}\}$, depicted by the red vertices in Figure 9 is a power dominating set of G . It is clear

that $S_1 = N[S] = \{v_1, v_4, v_7, v_8, v_9, v_{10}, v_{13}, v_{16}\}$. Therefore, all the vertices of S_1 are colored in red or green in Figure 9. The only unobserved neighbors of v_1, v_7, v_{16} and v_{10} are v_2, v_6, v_{11} and v_{15} respectively which will be observed in timestep 2. Next, v_{14} and v_{12} will be observed in timestep 3 as they are the only unobserved neighbors of v_{11} and v_{15} in timestep 2. Finally, v_5 and v_3 are observed in timestep 4 as they are the only unobserved neighbors of v_2 and v_6 in timestep 3 respectively. $\square$

Theorem 28. *If G is the Zero-symmetric graph then $\gamma_P(G) = 2$.*

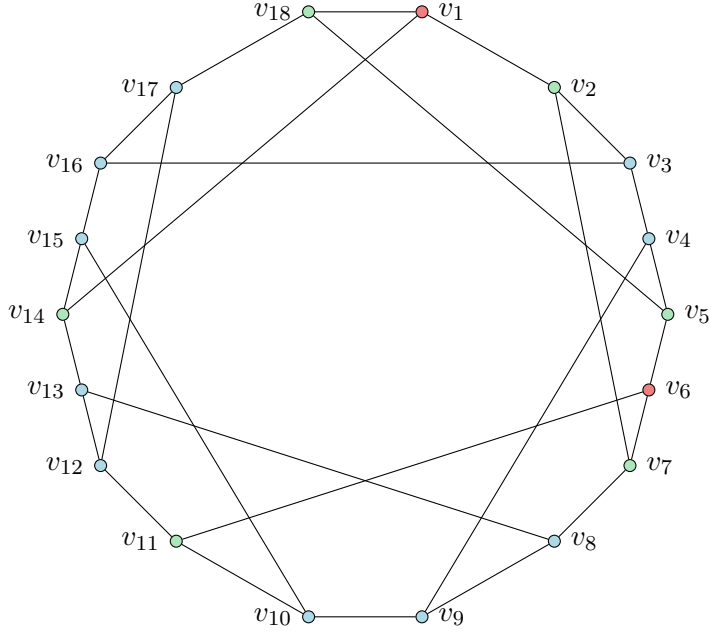


Figure 10: Zero-symmetric Graph

Proof. Since $g(G) > 4$, $\gamma_P(G) > 1$ by Lemma 2. We show that $S = \{v_1, v_6\}$, depicted by the red vertices in Figure 10 is a power dominating set of G . It is clear that $S_1 = N[S] = \{v_1, v_2, v_5, v_6, v_7, v_{11}, v_{14}, v_{18}\}$. Therefore, all the vertices of S_1 are colored red or green in Figure 10. The only unobserved neighbors of v_2, v_5, v_{18} and v_7 are v_3, v_4, v_{17} and v_8 respectively which will be observed in timestep 2. Next, v_{16} and v_9 will be observed in timestep 3 as they are the only unobserved neighbors of v_3 and v_4 in timestep 2. Finally, v_{15}, v_{10}, v_{12} and v_{13} are observed in timestep 4 as they are the only unobserved neighbors of v_{16}, v_9, v_{17} and v_8 in timestep 3 respectively. $\square$

Conjecture 1. *For a graph G with LCF notation $[5, -5]^n$ for $n \geq 10$, $\gamma_P(G) = 2$.*

4 Mycielski Graphs

We recall the famous Mycielski construction developed by Mycielski [5] in search of triangle-free graphs with large chromatic number. Given a graph G with $V(G) = \{v_1, \dots, v_n\}$, the Mycielskian of G , denoted $\mu(G)$, is the graph with vertex set $V(G) \cup V' \cup \{w\}$, where $V' = \{u_i \mid v_i \in V(G)\}$, and the edge set $E(G) \cup \{u_i v_j \mid v_i v_j \in E(G)\} \cup \{w u_j \mid u_j \in V'\}$. Note that G is a subgraph of $\mu(G)$. $\mu(P_4)$ is depicted in Figure 11. In this section, we study the power domination number of $\mu(G)$ when G is either a path, cycle, star, a complete graph, or wheel.

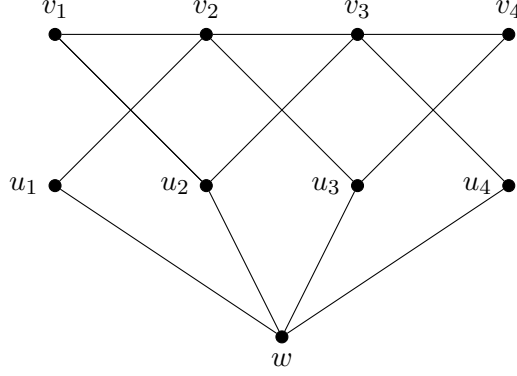


Figure 11: $\mu(P_4)$

Theorem 29. For $n \geq 2$, $\gamma_P(\mu(P_n)) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{2} \\ 2 & \text{if } n \equiv 1 \pmod{2} \end{cases}$

Proof. Suppose $V(P_n) = \{v_1, \dots, v_n\}$. Then $\mu(P_n)$ will have vertices $\{v_1, \dots, v_n\} \cup \{u_1, \dots, u_n\} \cup \{w\}$ such that for every $1 < i < n$ u_i is adjacent to w , v_i is adjacent to each of $u_{i+1}, u_{i-1}, v_{i+1}, v_{i-1}$, v_1 is adjacent to both u_2 and v_2 , and v_n is adjacent to both u_{n-1} and v_{n-1} .

Suppose first that $n \equiv 0 \pmod{2}$. We show that $S = \{w\}$ is a power dominating set of $\mu(P_n)$. Then $S_1 = N[S] = \{w, u_1, \dots, u_n\}$ because every u_j is adjacent to w for $1 \leq j \leq n$. Now since u_1 is adjacent to each vertex of $\{v_2, w\}$ and u_n is adjacent to each vertex of $\{v_{n-1}, w\}$ and both v_2 and v_{n-1} are the only unobserved neighbors of u_1 and u_n respectively, v_2 and v_{n-1} will be observed in timestep 2. Moreover, at timestep $i - 1$ for $2 \leq i \leq \lceil n/4 \rceil + 1$, v_{2i-2} is the only unobserved neighbor of u_{2i-3} . Thus, $v_{2i-2} \in S_i$. Similarly, at timestep $i - 1$ for $1 \leq i \leq \lceil n/4 \rceil + 1$, $v_{n-(2i-3)}$ is the only unobserved neighbor of $u_{n-(2i-4)}$ and $v_{n-(2i-3)} \in S_i$. If $n \equiv 0 \pmod{4}$, then the only unobserved vertices in $\mu(P_n)$ at timestep $\lceil n/4 \rceil + 1$ are of the form v_i where i odd and $i < n/2$ or v_i where i even and $i > n/2$. At timestep $\lceil n/4 \rceil + j - 1$ for $2 \leq j \leq n/4 + 1$, $v_{n/2+(2j-2)}$ is the only unobserved neighbor of $u_{n/2+(2j-3)}$.

Similarly, at timestep $\lceil n/4 \rceil + j - 1$ for $2 \leq j \leq n/4 + 1$, $v_{n/2-(2j-3)}$ is the only unobserved neighbor of $u_{n/2-(2j-4)}$. It follows that all vertices of $\mu(P_n)$ are observed by timestep $n/4 + n/4 + 1 = n/2 + 1$. If $n \equiv 2 \pmod{4}$, then the only unobserved vertices in $\mu(P_n)$ at timestep $\lceil n/4 \rceil + 1$ are of the form v_i where i odd and $i < n/2$ or v_i where i even and $i > n/2 + 1$. At timestep $\lceil n/4 \rceil + j - 1$ for $2 \leq j \leq n/4 + 1$, $v_{n/2+(2j-1)}$ is the only unobserved neighbor of $u_{n/2+(2j-2)}$. Similarly, at timestep $\lceil n/4 \rceil + j - 1$ for $2 \leq j \leq n/4 + 1$, $v_{n/2-(2j-2)}$ is the only unobserved neighbor of $u_{n/2-(2j-3)}$. Again, it follows that all vertices of $\mu(P_n)$ are observed by timestep $n/2 + 1$.

Next, suppose $n \equiv 1 \pmod{2}$. We claim that $S = \{v_1, w\}$ is a power dominating set of $\mu(P_n)$. Then $S_1 = N[S] = \{v_1, v_2\} \cup \{w, u_1, \dots, u_n\}$. Notice that the only unobserved neighbor of v_2 is v_3 so v_3 will be observed at timestep 2. Moreover, v_i will be the only unobserved neighbor of v_{i-1} at timestep $i - 2$ for $3 \leq i \leq n$. Thus, all vertices of $\mu(P_n)$ will be observed by timestep $n - 2$.

Now we will show that $\gamma_P(\mu(P_n)) \neq 1$ for $n \equiv 1 \pmod{2}$. Suppose, for the sake of contradiction, that $\gamma_P(\mu(P_n)) = 1$ and let $S = \{x\}$ be a power dominating set. If $w = x$, then $S_1 = N[S] = \{w, u_1, \dots, u_n\}$. As in the case of $n \equiv 0 \pmod{2}$, v_2 is the only unobserved neighbor of u_1 and v_{n-1} is the only unobserved neighbor of u_n . Thus, $\{v_2, v_{n-1}\} \subset S_2$. Moreover, at timestep $i - 1$ for $2 \leq i \leq \frac{n+1}{2}$, either v_{2i-2} is already observed or v_{2i-2} is the only unobserved neighbor of u_{2i-3} . Thus, all vertices of the form v_i where i even will be observed. This says that for each u_i where i odd, $N[u_i]$ is observed. Yet for each u_i where i even, there are two unobserved vertices in $N[u_i]$. Thus, $S = \{w\}$ is not a power dominating set. If $u_i = x$ for $1 < i < n$, then v_{i+1}, v_{i-1} , and w are observed but all three of these vertices are adjacent to at least two unobserved vertices. Thus, $u_i \neq x$ for $1 < i < n$ and a similar argument shows $u_i \neq x$ for $i \in \{1, n\}$. If $v_i = x$ for $1 < i < n$, then $u_{i+1}, u_{i-1}, v_{i+1}, v_{i-1}$ are observed but all four of these vertices are adjacent to at least two unobserved vertices. Thus, $v_i \neq x$ for $1 < i < n$ and a similar argument shows $v_i \neq x$ for $i \in \{1, n\}$. Hence, no such set S exists and $\gamma_P(\mu(P_n)) = 2$. □

Theorem 30. For $n \geq 4$, $\gamma_P(\mu(C_n)) = 2$.

Proof. Suppose C_n has vertices labelled $\{v_1, \dots, v_n\}$. Then $\mu(C_n)$ will have vertices $\{v_1, \dots, v_n\} \cup \{u_1, \dots, u_n\} \cup \{w\}$ such that every u_i is adjacent to w and v_i is adjacent to every u_{i+1} and u_{i-1} for $1 \leq i \leq n$.

We show that $S = \{v_1, w\}$ is a power dominating set of $\mu(C_n)$. $S_1 = N[S] = \{v_1, v_2, v_n\} \cup \{u_1, \dots, u_n\}$ because v_1 is adjacent to v_2 and v_n and every u_j is adjacent to w for $1 \leq j \leq n$. Notice that v_3 is the only unobserved neighbor of u_2 and v_4 is the only unobserved neighbor of u_3 at timestep 1. Thus, v_3 and v_4 will be observed in timestep 2. In fact, at timestep $\lfloor i/2 \rfloor$ for even i , the only unobserved neighbor of u_i is v_{i+1} and the only unobserved neighbor of u_{i+1} is v_{i+2} . Thus, all vertices of $\mu(C_n)$ will be observed by timestep $\lfloor n/2 \rfloor$.

Now we will show that $\gamma_P(\mu(C_n)) \neq 1$. Suppose, for the sake of contradiction, $\gamma_P(\mu(C_n)) = 1$ and $S = \{x\}$ is a power dominating set of $\mu(C_n)$. If $w = x$, then each vertex of $\{u_1, \dots, u_n\}$ is observed but every u_i is adjacent to v_{i+1} and v_{i-1} for $1 \leq i \leq n$ both of which are unobserved. If $u_i = x$ for $1 \leq i \leq n$, then v_{i+1}, v_{i-1} , and w are observed but all three of these vertices are adjacent to at least two unobserved vertices since $n \geq 4$. If $v_i = x$ for $1 \leq i \leq n$, then $u_{i+1}, u_{i-1}, v_{i+1}$, and v_{i-1} are observed but all four of these vertices are adjacent to at least two unobserved vertices. Hence, we have a contradiction. $\square$

Theorem 31. For $n \geq 2$, $\gamma_P(\mu(K_n)) = 1$.

Proof. Suppose $V(K_n) = \{v_1, \dots, v_n\}$. Then $\mu(K_n)$ will have vertices $\{v_1, \dots, v_n\} \cup \{u_1, \dots, u_n\} \cup \{w\}$ such that every u_i is adjacent to w and v_i is adjacent to every u_j for $1 \leq i, j \leq n$ and $j \neq i$.

We show that $S = \{v_1\}$ is a power dominating set of $\mu(K_n)$. $S_1 = N[S] = \{v_1, \dots, v_n\} \cup \{u_2, \dots, u_n\}$ because v_1 is adjacent to every v_i and every u_i for $1 < i \leq n$. Therefore, the only unobserved vertices at timestep 1 in $\mu(K_n)$ would be w and u_1 . Since v_2 is adjacent to u_1 but not w and u_2 is adjacent to w but not u_1 , then both w and u_1 will be observed in timestep 2. Thus, all vertices of $\mu(K_n)$ are observed by timestep 2. $\square$

Theorem 32. For $n \geq 1$, $\gamma_P(\mu(K_{1,n})) = 1$.

Proof. Suppose $V(K_{1,n}) = \{v_1, \dots, v_{n+1}\}$ where v_t is the universal vertex of $K_{1,n}$. Then $\mu(K_{1,n})$ will have vertices $\{v_1, \dots, v_{n+1}\} \cup \{u_1, \dots, u_{n+1}\} \cup \{w\}$ such that every u_i is adjacent to w and v_t is adjacent to every u_j and v_j for $1 \leq i, j \leq n+1$ and $i, j \neq t$.

We show that $S = \{v_t\}$ is a power dominating set of $\mu(K_{1,n})$. $S_1 = N[S] = \{v_1, \dots, v_{n+1}\} \cup \{u_1, \dots, u_{n+1}\} - \{u_t\}$ because v_t is adjacent to every v_j and every u_j for $1 \leq j \leq n+1, j \neq t$. Therefore, the only unobserved vertices in $\mu(K_{1,n})$ would be w and u_t . Since, for $a \neq t$, v_a is adjacent to u_t but not w and u_a is adjacent to w but not u_t , then w and u_t will be observed in time step 2. Thus, all vertices of $\mu(K_{1,n})$ are observed by timestep 2. $\square$

Theorem 33. For $n \geq 4$, $\gamma_P(\mu(W_n)) = 1$.

Proof. Suppose $V(W_n) = \{v_1, \dots, v_n\}$ where v_t is the universal vertex of W_n . Then $\mu(W_n)$ will have vertices $\{v_1, \dots, v_n\} \cup \{u_1, \dots, u_n\} \cup \{w\}$ such that every u_i is adjacent to w and v_t is adjacent to every u_j for $1 \leq j \leq n$ and $j \neq t$.

We show that $S = \{v_t\}$ is a power dominating set of $\mu(W_n)$. $S_1 = N[S] = \{v_1, \dots, v_n\} \cup \{u_1, \dots, u_n\} - \{u_t\}$ because v_t is adjacent to every v_i for $i \neq t$ and every u_j for $1 \leq j \leq n$ where $j \neq t$. Therefore, the only unobserved vertices in $\mu(W_n)$ will be $\{w\}$ and $\{u_t\}$. Since v_i for $i \neq t$ is adjacent to u_t but not w and u_i is adjacent to w but not u_t , then $\{w\}$ and $\{u_t\}$ will be observed in timestep 2. Thus, all vertices of $\mu(W_n)$ are observed by timestep 2. $\square$

Open Questions:

1. Suppose G is a cubic graph with LCF notation $[k, -k]^n$ and H is a cubic graph with LCF notation $[k, -k]^m$. Is it true that $\gamma_P(G) = \gamma_P(H)$?
2. Are there classes of graphs, for which the $\gamma_P(G) \leq sp(T)$ where T is a spanning tree of G ?

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